Problem Three of Chapter Eight of the *Nine Chapters*

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Problem Three of Chapter Eight of the *Nine Chapters* states that, “None of the yields of two bundles of the best grain, three bundles of ordinary grain, and four bundles of the worst grain are sufficient to make a whole measure. If we add to the good grain one bundle of the ordinary, to the ordinary one bundle of the worst, and to the worst one bundle of the best, then each yield is exactly one measure. How many measures does one bundle of each of the three types of grain contain?” By solving problems as such, the Chinese were able to solve complicated problems by representing them mathematically. Their method of solving systems of equations was called the Chinese Elimination Method and was used to set up modern ideas associated with Linear Algebra.

Mathematical Representation

After reading this problem, there were a few assumptions that we had to make. For one, this translation is not a perfect one due to multiple words having the same meaning in Chinese, but not in English. The words “good” and “great” have no difference in meaning in Chinese, but obviously have different meanings in English. Once we understood the language differences, we were able to develop a system of three linear equations with a total of three unknown variables. Then we could solve the system of equations using the Chinese Elimination Method. Our equations for the mathematical representation of this problem are as follows,

$$2B + D + 0W = 1 \qquad 0B + 3D + W = 1 \qquad B + 0D + 4W = 1$$

Where B is defined as the Best type of grain, D represents the Ordinary type of grain, and W represents the Worst type of grain. The coefficients in front of these variables are how many bundles of that grain we have. For example, in the problem it says, “If we add...to the ordinary one bundle of the worst...then [the] yield is exactly one measure.” From this, we were able to formulate the second equation, $0B + 3D + W = 1$. Now we can represent these linear equations as a matrix by taking the coefficients in front of each of the variables and writing them down in a vertical column, with the last number in the column being the answer to that particular equation. A matrix is a way to organize linear equations in a rectangular array that allows us to solve a system of simultaneous equations for each variable. In the context of the problem, it allows us to solve for the amount of measures one bundle of each of the three types of grain contains. Our linear equations look as follows in the form of a matrix,

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

where each value in the columns represent the coefficients on the types of grain in our system of equations as you read down the column. The purpose of using Chinese Elimination is to eliminate a column until it becomes an equation of one variable. We can tell a column is an equation of a single variable if two of the four elements in the column are zeroes. The formula the Chinese used to eliminate columns down to a single variable is as follows,

$$a_n\, C_i - a_i\, C_n = C_i$$

where a_n is a number in the column C_n which is used to replace the column C_i by using a number in that column, a_i. In other words, we want to replace columns that contain equations of multiple variables or unknowns and turn them into columns that contain single variable equations that we can solve using Algebra. We will work through the columns from right to left as the Chinese did in this technique. Starting with our first matrix in the first row, we would like to call our $a_n = 2$ and use that to eliminate the $a_i = 1$ (which is the first column). Plugging into the formula above yields the following expression,

$$\begin{matrix} \boxed{1} & 0 & \boxed{2} \\ 0 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$2\begin{pmatrix} 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 8 \\ 1 \end{pmatrix}$$

where the equivalent matrix replaces the first column in the original matrix. We can see from this that the Chinese Elimination Method was successful since it eliminated $a_i = 1$ and replaced it with a zero. After replacing that column, our new matrix looks like,

$$\begin{matrix} 0 & 0 & 2 \\ -1 & \boxed{3} & \boxed{1} \\ 8 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

where we can perform Chinese Elimination again by using the 1 in the third column to replace the 3 in the second column.

$$1\begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

After performing Chinese Elimination again, we can see that the resulting matrix was a success in that it replaced the 3 in the second column with a zero, which is what we should expect going forward. Now, our resulting matrix with the second column replaced is as follows.

$$\begin{array}{ccc} 0 & \boxed{-6} & \boxed{2} \\ -1 & 0 & 1 \\ 8 & 1 & 0 \\ 1 & -2 & 1 \end{array}$$

Now we can perform Chinese Elimination again in hopes of getting a column with 2 zeroes.

$$2\begin{pmatrix} -6 \\ 0 \\ 1 \\ -2 \end{pmatrix} - (-6)\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 2 \\ 2 \end{pmatrix}$$

Now our new matrix after replacing the middle column.

$$\begin{array}{ccc} 0 & 0 & 2 \\ \boxed{-1} & \boxed{6} & 1 \\ 8 & 2 & 0 \\ 1 & 2 & 1 \end{array}$$

We can perform the Chinese Elimination Technique one last and final time by replacing the first column by using the middle column.

$$6\begin{pmatrix} 0 \\ -1 \\ 8 \\ 1 \end{pmatrix} - (-1)\begin{pmatrix} 0 \\ 6 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 50 \\ 8 \end{pmatrix}$$

Now we can write our final matrix with the first row replaced to have 2 zeroes.

$$\begin{array}{ccc} 0 & 0 & 2 \\ 0 & 6 & 1 \\ 50 & 2 & 0 \\ 8 & 2 & 1 \end{array}$$

From this matrix we can write out three new linear equations just as we did when we put equations into the matrix. Our three new equations are as follows.

$0B + 0D + 50W = 8$ $\qquad$ $0B + 6D + 2W = 2$ $\qquad$ $2B + 1D + 0W = 1$

Using the first equation, we can solve for the variable “W” which we can use to solve for the other variables using the other two equations. Dividing by “50” on both sides of the first equation yields the result W= $\frac{8}{50}$. Plugging that into our second equations yields the equation.

$$6D + 2\left(\frac{16}{50}\right) = 2$$

Subtracting ($\frac{32}{50}$) from both sides of the equation and then dividing both sides of the equation by 6 yields the result that D= $\frac{84}{300}$. Substituting that value of "D" into the final equation yields the result.

$$2B + \frac{84}{300} = 1$$

Subtracting ($\frac{84}{300}$) from both sides of the equation and then dividing by two gives the result that B= $\frac{9}{25}$.

Once we eliminated enough columns in the matrix that led to a single column holding one variable, we were able to use this information and solve for the rest of the variables. Establishing a solution to the rest of the variables provided the total amount of measures one bundle held. Looking at the solutions derived from the matrices, the Best type of grain holds $\frac{9}{25}$ of a whole bundle, the Ordinary grain contains $\frac{84}{300}$ of a whole bundle, and the Worst grain holds $\frac{8}{50}$ of a whole bundle. Which when these solutions to the three grains are summed together, produce an answer less than one. Therefore, proving "None of the yields of two bundles of the best grain, three bundles of ordinary grain, and four bundles of the worst grain are sufficient to make a whole measure."