CMSC 208: Homework 1

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1. Prove using Induction:

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

base case:

$$\sum_{i=1}^{1} 1^3 = \frac{1}{4}1^2(1+1)^2$$
$$1 = \frac{1}{4} \cdot 1 \cdot 4$$
$$= \frac{4}{4}$$
$$= 1$$

assume (inductive hypothesis):

$$\sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2$$

want to prove:

$$\sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

$$\sum_{i=1}^{k+1} i^3 = (k+1)^3 + \sum_{i=1}^{k}$$
$$= (k+1)^3 + \frac{k^2(k+1)^2}{4}$$
$$= \frac{4(k+1)^3}{4} + \frac{k^2(k+1)^2}{4}$$
$$= \frac{4(k+1)^3 + k^2(k+1)^2}{4}$$
$$= \frac{(k+1)^2(4(k+1) + k^2}{4}$$
$$= \frac{(k+1)^2(4k+4+k^2)}{4}$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$
$$= \frac{(k+1)^2((k+1)+1)^2}{4}$$

2. Prove or disprove:

   (a) the sum of any three consecutive integers is even

   Disproven by counter example:
   $2 + 3 + 4 = 9$
   9 is an odd number, therefore the above statement is disproven

   (b) the product of any three consecutive integers is even

odd-even-odd case	**even-odd-even case**
1 * 2 * 3 = 6	2 * 3 * 4 = 24
5 * 6 * 7 = 210	6 * 7 * 8 = 336
9 * 10 * 11 = 990	10 * 11 * 12 = 1320
-1 * 0 * 1 = 0	-2 * -1 * 0 = 0

   The above table shows multiple examples for each case, which proves the above statement.

3. Prove using pigeonhole principle
   For this example, let S be the set of students' grades

   S = {67, 72, 83, 94, 51, 70, 81}

   Below is a mock hash table with each pair's sum and difference modded by 10

<table>
<tr><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th></tr>
<tr><td>67+83</td><td>67+94</td><td>|72-94|</td><td>|67-70|</td><td>|67-81|</td><td>|67-72|</td><td>|67-83|</td><td>|67-94|</td><td>67+51</td><td>67+72</td></tr>
<tr><td>|51-81|</td><td>|72-83|</td><td>72+70</td><td>72+51</td><td>83+51</td><td>72+83</td><td>|67-51|</td><td>67+70</td><td>67+81</td><td>|72-81|</td></tr>
<tr><td></td><td>|72-51|</td><td>|72-70|</td><td>72+81</td><td>83+81</td><td>94+51</td><td>72+94</td><td>83+94</td><td></td><td>|51-70|</td></tr>
<tr><td></td><td>|83-94|</td><td>|83-51|</td><td>83+70</td><td>94+70</td><td>94+81</td><td></td><td></td><td></td><td></td></tr>
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<tr><td></td><td>70+81</td><td>51+81</td><td>94-51</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>|70-81|</td><td></td><td>94-81</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

In any set of 7 positive integers, you are guaranteed to have at least two values whose values in the ones place sums to 10, or the values in the ones place are the same. By the Pigeonhole Principle, any set of seven positive integers will always have one pair of values whose sum or difference are divisible by 10.

4. Compute

(a)

$$\sum_{i=1}^{10}(-1)^i i$$

$$\begin{aligned}\sum_{i=1}^{10}(-1)^i i &= (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + \cdots + (-1)^{10} \cdot 10 \\ &= (-1) + 2 + (-3) + 4 + (-5) + 6 + (-7) + 8 + (-9) + 10 \\ &= ((-1) + 2) + ((-3) + 4) + ((-5) + 6) + ((-7) + 8) + ((-9) + 10) \\ &= 1 + 1 + 1 + 1 + 1 \\ &= 5\end{aligned}$$

(b)

$$\sum_{i=1}^{1000}(-1)^i i$$

using the same pattern from part a, this sum computes to 500
there are 500 pairs of numbers that will sum to 1 (all odds are negative, and all evens positive)

5. Explain the following symbolic expression in English

$$\mathbb{E}' = \{n \in \mathbb{N} : \text{n is even and } n > 2\}$$

$$\mathbb{P} = \text{the set of primes}$$

$$\forall n \in \mathbb{E}', \exists p_1, p_2 \in \mathbb{P}, p_1 + p_2 = n$$

for all even numbers greater than 2 (n), there exists two prime numbers (p_1 and p_2) that will sum to said even number ($p_1 + p_2 = n$)

6. Prove for $x > -1$:

$$\forall n \in \mathbb{N}, (1+x)^n \geq 1 + nx$$

base case:

$$(1+x)^1 \geq 1 + (1)x$$
$$1 + x \geq 1 + x$$

Assume (inductive hypothesis):

$$(1+x)^k \geq 1 + kx$$

Prove:

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

$$\begin{aligned} (1+x)^{k+1} = (1+x)^k \cdot (1+x)^1 &\geq 1 + (k+1)x \\ &\geq (1+kx)(1+x) \\ &\geq 1 + (x + kx) + k(x)^2 \\ &\geq 1 + x(1+k) + k(x)^{2*} \end{aligned}$$

*: we know that x is greater than -1, therefore, x will never be less than zero

$$1 + (k+1)x + k(x)^2 \geq 1 + (k+1)x$$