Honors Project

Annuities, Amortizations, Credit Cards and Financial Simulations

MATH 461

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Part One Annuities

In order to observe the effects of interest and time on investments we will look at an example of a set of twins saving for retirement. The twins, Abe and Ben, use the same bank, so they receive the same interest rate of 8% compounded monthly for the entire time their investments are held. Note that the model assumes that there is no money pulled out of the investments at any time and that the payments and interest are invested at the end of the month. Hence, the monthly balance can be described with the following recurrence relations:

$$a_{n+1} = a_n + interest + deposit$$

Abe decides to open his account at 20 years old, and invest $200 a month until the age of thirty. However, the investment is still held and continues to receive the 8% interest compounded monthly.

Ben does not open his account till he is 30 years old, and invest $200 a month until the age of 65 years old. At age 65, both brothers retire and begin to live off of their retirement money (hence investments have ceased).

Below in Figure One, is the first 25 months of the model. In Figure Two, is the last 25 months of the model, again with an interest rate of 8% compounded monthly.

"Monthly Investment"	200.00	
"Rate"	0.0800	
Month	Abe	Ben
1	200.00	0.00
2	401.33	0.00
3	604.01	0.00
4	808.04	0.00
5	1013.42	0.00
6	1220.18	0.00
7	1428.31	0.00
8	1637.84	0.00
9	1848.75	0.00
10	2061.08	0.00
11	2274.82	0.00
12	2489.99	0.00
13	2706.59	0.00

14	2924.63	0.00
15	3144.13	0.00
16	3365.09	0.00
17	3587.52	0.00
18	3811.44	0.00
19	4036.85	0.00
20	4263.76	0.00
21	4492.19	0.00
22	4722.13	0.00
23	4953.61	0.00
24	5186.64	0.00
25	5421.22	0.00

Figure One

515	504893.58	383969.2748
516	508259.54	386729.07
517	511647.94	389507.26
518	515058.92	392303.98
519	518492.65	395119.3387
520	521949.27	397953.47
521	525428.93	400806.49
522	528931.79	403678.53
523	532458.00	406569.7242
524	536007.72	409480.19
525	539581.11	412410.06
526	543178.31	415359.46
527	546799.50	418328.5204
528	550444.83	421317.38
529	554114.46	424326.16
530	557808.56	427355.00
531	561527.28	430404.0341
532	565270.80	433473.39
533	569039.27	436563.22
534	572832.87	439673.64
535	576651.75	442804.796
536	580496.10	445956.83
537	584366.07	449129.87
538	588261.85	452324.07
539	592183.59	455539.5665
540	596131.48	458776.50

Figure Two

As seen in Figure Two, in the last row, Abe's ending balance was $596,131.48 and Ben's end balance was $458,776.50. However, Abe had only invested $24,000 (120 months X $200 a month = $24,000) into his account, and Ben invested $72,000 (360 months X $200 a month = $72,000) into his account. Although, by changing the interest in the model, it can be found that if a lower interest rate was used Ben would have had a larger final balance. Below in Figure Three, were the different rates tested. By rounding to the nearest hundredth percent, it was found that the highest interest rate where Ben has a larger total was 6.39%. Now, if the brothers decided that they both wanted to have at least $1,000,000, then they could choose to open an investment with a higher interest rate. Seen in Figure Four, they would need to open an investment with an interest rate of at least 11.06%

Interest Rate	Abe's Total	Ben's Total	Ben -Abe
0.01	35797.82	100526.58	64728.76
0.02	53421.78	121509.56	68087.78
0.03	79761.86	148312.73	68550.87
0.04	119147.76	182746.19	63598.43
0.05	178068.97	227218.49	49149.52
0.06	266256.46	284942.06	18685.6
0.063	300437.55	305450.87	5013.32
0.0635	306547.91	309031.14	2483.23
0.0637	309026.82	310476.67	1449.85
0.0638	310273.8	311202.34	928.54
0.0639	311525.82	311929.94	404.12
0.064	312782.92	312659.5	-123.42
0.065	325637.12	320063.24	-5573.88
0.07	398308.02	360210.92	-38097.1
0.08	596131.48	458776.5	-137355

Figure Three

Interest Rate	Abe's Total	Ben's Total
0.1	1337172.73	759327.61
0.11	2004023.38	985659.27
0.1105	2045001.19	998742.24
0.1106	2053297.02	1001381.17
0.111	2086819.18	1012012.07
0.121	3129022.29	1321237.14
0.125	3679851	1471917
0.12	3004746.72	1286191.89

Figure Four

Part Two Amortizations

Using Excel, a model was made to analyze an Amortization Schedule for a mortgage of $125,000 for 30 years at 5.9% annual interest compounded monthly. Below, in Figure Five, the first twelve months can be seen. Note that after one year the balance is $124,873.16. In addition, the twelve payments sum to $8,897.02, but only $1563.89 went towards the reduction of the principal.

Payment Number	Payment Amount	Interest for Period	Reduction of Principal	Unpaid Balance
0				125000.00
1	741.42	614.58	126.84	124873.16
2	741.42	613.96	127.46	124745.70
3	741.42	613.33	128.09	124617.61
4	741.42	612.70	128.72	124488.89
5	741.42	612.07	129.35	124359.54
6	741.42	611.43	129.99	124229.55
7	741.42	610.80	130.62	124098.93
8	741.42	610.15	131.27	123967.66
9	741.42	609.51	131.91	123835.75
10	741.42	608.86	132.56	123703.19
11	741.42	608.21	133.21	123569.98
12	741.42	607.55	133.87	123436.11

Figure Five

An interest rate of 5.9% is good, and allows for affordable, reasonable payments. If we take the same mortgage, with the high interest rate from 1981 of 18.63% the monthly payment is raised to $1948.23. Below in Figure Six, is the payments for the first twelve months. Note that after one year the balance is $12,490.55 and $ 99.45 was paid toward the principal in the first year.

Payment Number	Payment Amount	Interest for Period	Reduction of Principal	Unpaid Balance
0				125000.00
1	1948.23	1940.63	7.60	124992.40
2	1948.23	1940.51	7.72	124984.68
3	1948.23	1940.39	7.84	124976.84
4	1948.23	1940.27	7.96	124968.88

5	1948.23	1940.14	8.09	124960.79
6	1948.23	1940.02	8.21	124952.58
7	1948.23	1939.89	8.34	124944.24
8	1948.23	1939.76	8.47	124935.77
9	1948.23	1939.63	8.60	124927.17
10	1948.23	1939.49	8.74	124918.43
11	1948.23	1939.36	8.87	124909.56
12	1948.23	1939.22	9.01	124900.55

Figure Six

Note that in the earlier years of paying a mortgage, interest is the majority of what is being paid off. Below, in Figure Seven are the results of paying an extra one hundred dollars in the first two months using the same model. As a result, the balance after a year is $124665.42, $334.58 was paid towards the principle, and it take 282 months to pay off the mortgage compared to 360 months.

Payment Number	**Payment Amount**	**Interest for Period**	**Reduction of Principal**	**Unpaid Balance**
0				125000.00
1	2048.23	1940.63	107.60	124892.40
2	2048.23	1938.95	109.28	124783.12
3	1948.23	1937.26	10.97	124772.15
4	1948.23	1937.09	11.14	124761.01
5	1948.23	1936.91	11.32	124749.69
6	1948.23	1936.74	11.49	124738.20
7	1948.23	1936.56	11.67	124726.53
8	1948.23	1936.38	11.85	124714.68
9	1948.23	1936.20	12.03	124702.65
10	1948.23	1936.01	12.22	124690.43
11	1948.23	1935.82	12.41	124678.02
12	1948.23	1935.63	12.60	124665.42

Figure Seven

Next, we will discuss common misconceptions and patterns surrounding interest. In the first case, an individual claims that doubling the size of the mortgage will double the amount of your monthly payment. Below in Figure Eight and Nine, we can see the result of the first twelve payments towards a mortgage of $125,000 and $250,000. Note that the monthly payments are doubled, when the mortgage is doubled; this is a result of the interest for the period and the reduction for the principal being doubled.

Payment Number	Payment Amount	Interest for Period	Reduction of Principal	Unpaid Balance
0				125000.00
1	749.44	625.00	124.44	124875.56
2	749.44	624.38	125.06	124750.50
3	749.44	623.75	125.69	124624.81
4	749.44	623.12	126.32	124498.49
5	749.44	622.49	126.95	124371.54
6	749.44	621.86	127.58	124243.96
7	749.44	621.22	128.22	124115.74
8	749.44	620.58	128.86	123986.88
9	749.44	619.93	129.51	123857.37
10	749.44	619.29	130.15	123727.22
11	749.44	618.64	130.80	123596.42
12	749.44	617.98	131.46	123464.96

Figure Eight

Payment Number	**Payment Amount**	**Interest for Period**	**Reduction of Principal**	**Unpaid Balance**
0				250000.00
1	1498.88	1250.00	248.88	249751.12
2	1498.88	1248.76	250.12	249501.00
3	1498.88	1247.51	251.37	249249.63
4	1498.88	1246.25	252.63	248997.00
5	1498.88	1244.99	253.89	248743.11
6	1498.88	1243.72	255.16	248487.95
7	1498.88	1242.44	256.44	248231.51
8	1498.88	1241.16	257.72	247973.79
9	1498.88	1239.87	259.01	247714.78
10	1498.88	1238.57	260.31	247454.47
11	1498.88	1237.27	261.61	247192.86
12	1498.88	1235.96	262.92	246929.94

Figure Nine

Next, an individual claims that doubling the period of a mortgage halves the amount of your monthly payment. This is a misconception. By doubling the period one does reduce the monthly payments. For example, if an individual has a mortgage of 125,000 dollars with an APR of 6% for thirty years, then the monthly payment is $749.44. However, if the mortgage and APR stays the same and thirty years is extended to sixty, then the monthly payment is only reduced to $642.72, not $374.72. This is because more interest has to be paid due to the reduction of the principal taking a longer time.

The same individual claims that in order to retire on an income of $1000 per month for 20 years, then one needs to save $1000 per month for 20 years. Thus, the individual did not include the addition of interest into the savings account, and believed that the ending balance would equal $240,000, but this is not true. Consider the following example, if an individual opens a savings account for retirement by saving $1000 per month for 20 years with an APR of 6% compounded monthly, then after 20 years the ending balance is $ 464,351.10. This is a result of .005 of the balance being added into the account every month.

Part Three Amortizations with Calculus

Note that we can also analyze the different effects that cost, interest, and time have on a monthly payment using calculus. The monthly payment *P* on a mortgage loan of $*A* at an APR of *r* for *t* years is given by this formula:

$$P(A, r, t) = A * \frac{r/12}{1 - (1 + \frac{r}{12})^{-12t}}$$

Using substitution,

$$P(\$150{,}000, 7.5\%, 20) = 150{,}000 * \frac{.075/12}{1 - (1 + \frac{.075}{12})^{(-12*20)}}$$

$$P(\$150{,}000, 7.5\%, 20) = 1208.39$$

Thus, the monthly payment for a mortgage of $150,000 for twenty years at 7.5% annual interest compounded monthly is $1,208.39.

Note, that using differentials we can solve for the changes in payment if the loan amount, APR, or numbers of years increases, using the formula:

$$dP = \frac{dP}{dA} * dA + \frac{dP}{dr} * dr + \frac{dP}{dt} * dt$$

Note:

$$\frac{dP}{dA} = \frac{r/12}{1-(1+\frac{r}{12})^{-12t}}$$

$$\frac{dP}{dr} = A(-12^{-12t-1}) * (12t+1) * r^{12t}$$

$$\frac{dP}{dt} = A12^{-12t}r^{12t+1} * (\log(12) - \log(r))$$

Consider the following cases, the loan amount increased by 10%, the APR increased by 10%, and the number of years increased by 10%.

In Case One, the new loan would be $165,000 and the APR and length of time stay the same, thus the change in the monthly payment can be solved by:

$$dP = \frac{r/12}{1-(1+\frac{r}{12})^{-12t}} * (165000 - 150000) = 120.84$$

In order to find the exact change for Case One, we would solve for:

$$\Delta P = P(165{,}000, .075, 20) - P(150{,}000, .075, 20)$$

$$\Delta P = \left(165{,}000 * \frac{\frac{.075}{12}}{1-(1+\frac{.075}{12})^{(-12*20)}}\right) - \left(165{,}000 * \frac{\frac{.075}{12}}{1-(1+\frac{.075}{12})^{(-12*20)}}\right) = 120.84$$

For Case Two, the APR is raised to 8.25% and the mortgage and length of time stay the same, thus the change in the monthly payment can be solved by:

$$\frac{dP}{dr} = (A(-12^{-12t-1}) * (12t+1) * r^{12t}) * (.0825 - .075) = 69.71$$

In order to find the exact change for Case Two, we would solve for

$$\Delta P = P(150{,}000, .0825, 20) - P(150{,}000, .075, 20)$$

$$\Delta P = \left(150{,}000 * \frac{\frac{.0825}{12}}{1-(1+\frac{.0825}{12})^{(-12*20)}}\right) - \left(150{,}000 * \frac{\frac{.075}{12}}{1-(1+\frac{.0825}{12})^{(-12*20)}}\right) = 69.71$$

For Case Three, the number of years is increased to 22, and the mortgage and APR stay the same. Thus, the change in the monthly payment can be solved by:

$$\Delta P = A12^{-12t}r^{12t+1} * (\log(12) - \log(r)) * (22 - 20) = 46.52$$

In order to find the exact change for Case Three, we would solve for

$$\Delta P = P(150{,}000, .075, 22) - P(150{,}000, .075, 20)$$

$$\Delta P = \left(150{,}000 * \frac{\frac{.075}{12}}{1 - \left(1 + \frac{.075}{12}\right)^{(-12*22)}}\right) - \left(150{,}000 * \frac{\frac{.075}{12}}{1 - \left(1 + \frac{.075}{12}\right)^{(-12*20)}}\right)$$

$$\Delta P = -46.52$$

Therefore, raising the cost of the loan or the APR results in a higher monthly payment, with the loan having the biggest impact. Contrasting, by increasing the years by 10%, it reduced the monthly payment.

Part Four Credit Cards

Using Excel, we have modeled Dan's credit card debt. Dan has a credit card balance of $5,000 that charges 18% APR. He is required to make a minimum monthly payment that is the greater of $15 or 2% of the outstanding balance. In making the model, it was assumed that the monthly payment is processed on the last day of the month and that the billing cycle ends on the last day of the month and that he finance charge is calculated using the ending balance for the month. In addition, it was assumed that Dan paid every payment on time and was never charged a late fee. Dan thought that he could only pay the minimum as a result it took him 450 months or thirty-seven and a half years to pay off his credit card. Also, his payments over those 450 months added up to $ 17,673.77, thus Dan paid $12,673.77 in interest.

Part Five Financial Simulations

In order to compare and examine financial scenarios this next section focuses on three different simulations with a starting investment of $1,000.

The first simulation starts with an initial investment of $1,000, and the amount of interest received each year is a constant that varies uniformly between -20% and 20% for 25 years. The results of 20 simulations can be seen below in Figure Ten.

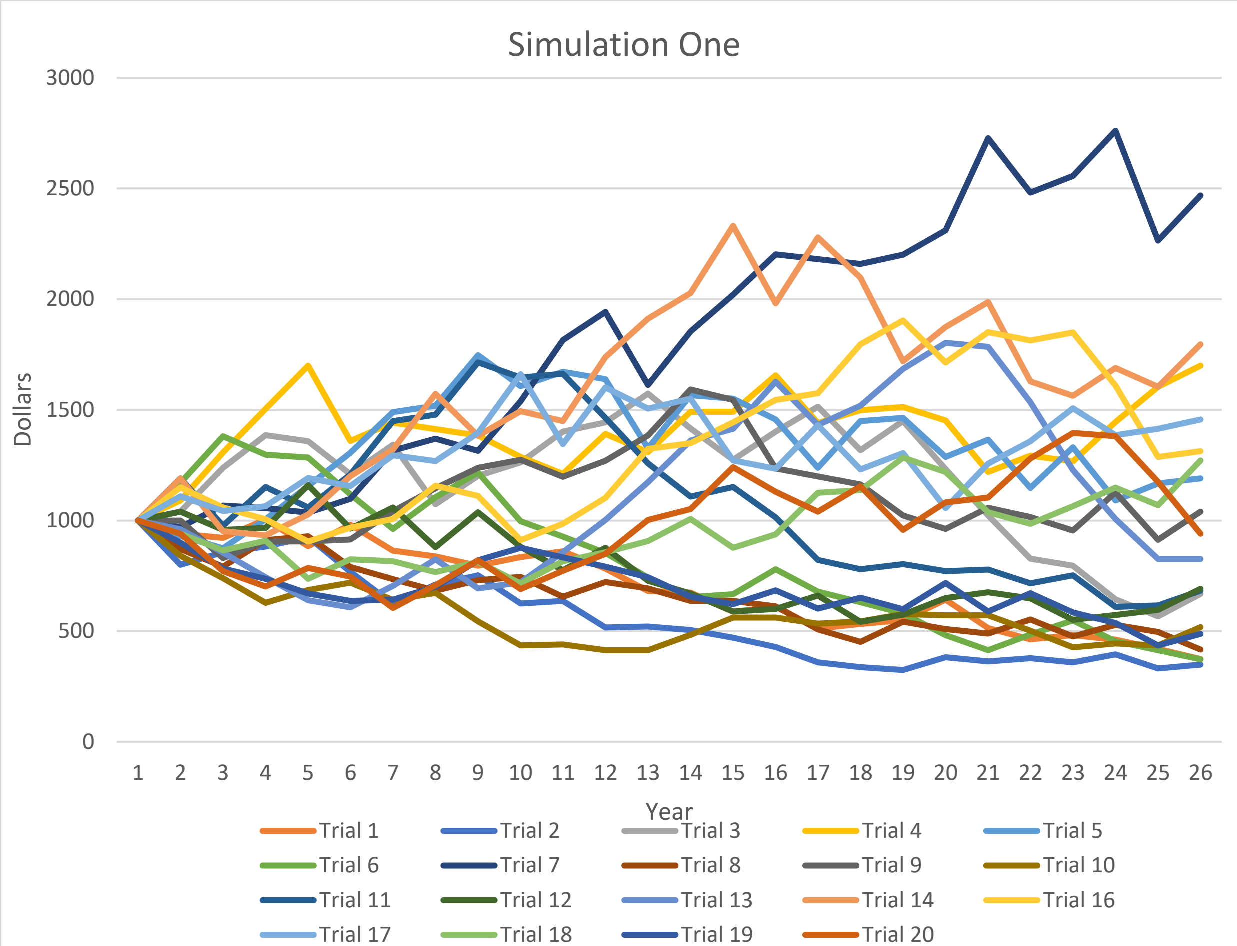

Figure Ten

In the second simulation, an initial investment of $1,000 dollars was made, and the first year's interest was randomly chosen using a uniform distribution between -20% and 20%. However, if the year previous interest rate was negative, the interest rate for the present year was randomly chosen using a uniform distribution between -20% and 10% for 25 years. If the year previous interest rate was positive, the interest rate for the present year was randomly chosen using a uniform distribution between -10% and 20% for 25 years. The 20 trials can be seen below in Figure Eleven.

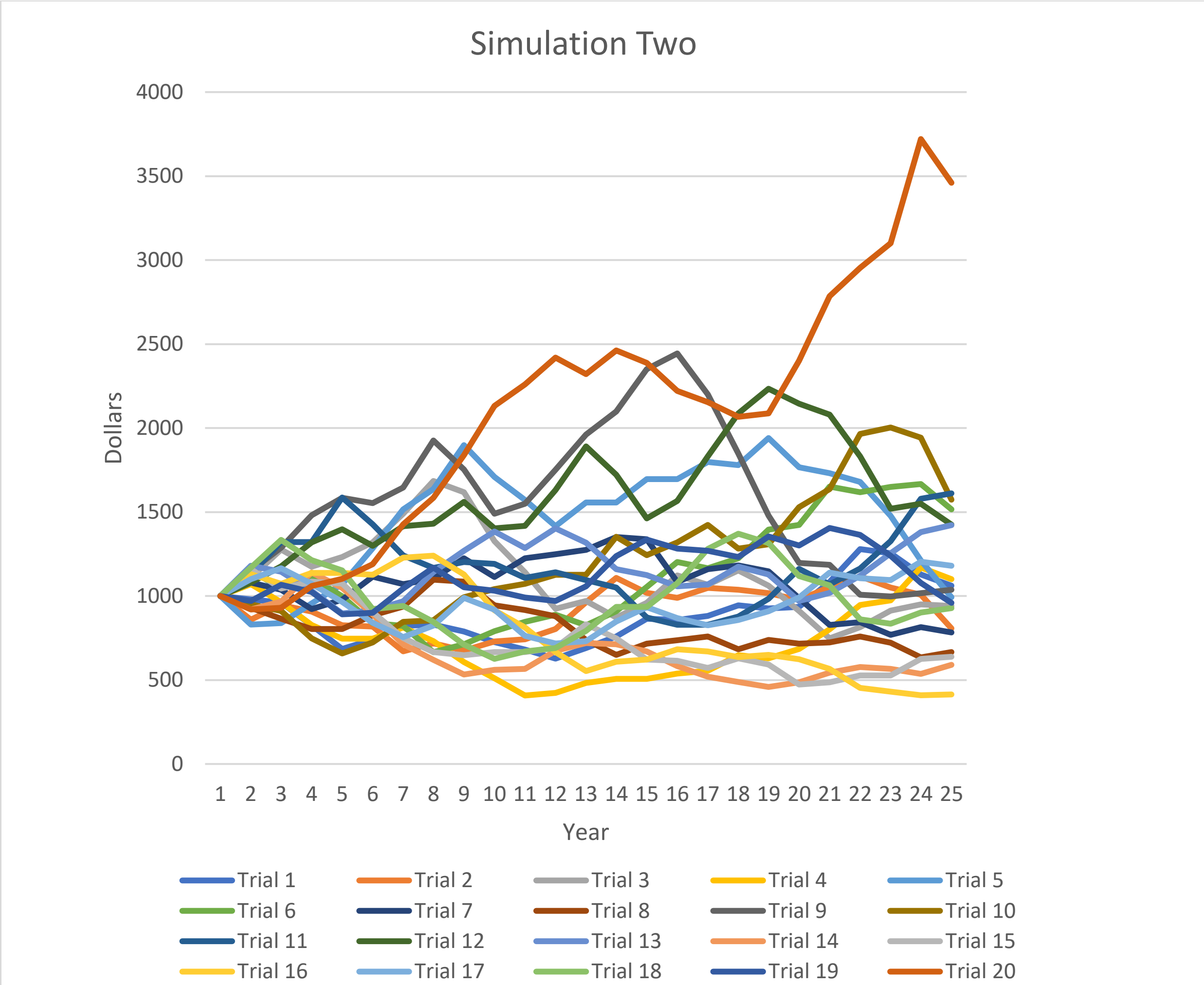

Figure Eleven

The third simulation starts with an initial investment of $1000 and the first year's interest was randomly chosen using a uniform distribution between -20% and 20%. However, the following month's interest rate is within 5 percentage points of the previous year's interest rate, randomly chosen using a uniform distribution for 25 years. However, it was assumed that the rate will always be between -30% and 30%. Below, in Figure Twelve, are the result of the twenty trials for the third simulation.

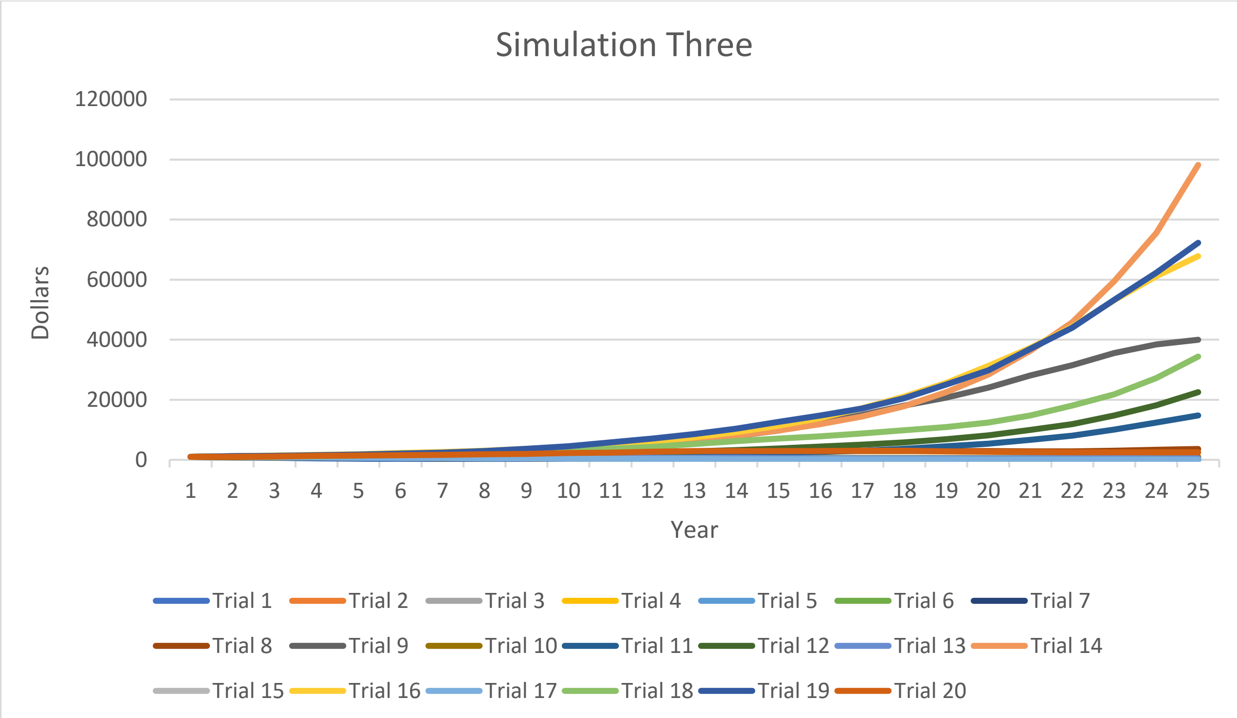

Figure Twelve

Note, The first simulation's ending balances had a minimum of $348.30 , an average of $954.48 , and a maximum of $2,468.21. The second simulation's ending balances had a minimum of $446.51 , an average of $111.43, and a maximum of $3183.47. The third simulation's ending balances had a minimum of $1.29 , an average of $21,178 , and a maximum of $126,680. By looking at the Figures Ten, Eleven and Twelve, it can be observed the third simulation had the biggest spread. Note the standard deviation was 574.22 for Simulation One was, was 595.47 for Simulation Two, and 35,515 for the Simulation Three.

Comparing the three simulations, the third simulation seems to be the most realistic, due to in dependency on the interest rate from the previous year. Looking at Figure Twelve, that shows the twenty trials from simulation three, it can be observed that if this model was an accurate representation of certain investments, that it could be beneficial to keep investments that have a similar pattern to Trial 2, Trial 11, and Trial 14.